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THE COMPRESSION OF AN UNBOUNDED BODY WITH SEMI-INFINITE CYLINDRICAL CAVITIES*

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The deformation of an unbounded elastic body with two semi-infinite cylindrical cavities (Fig.1) under uniform compression by a load p_∞ at infinity, perpendicular to the axis of symmetry is considered. An infinite cylinder $r \leqslant a$ and an infinite body with a hole of radius b are separated by surfaces $r = a$ and $r = b$ from the body mentioned. For $|z| < c$ the interval between them is filled with thin rings. For $a = 0$ a bottomless pit is obtained that is partially filled with discs of radius b; the case $b - a \ll b$ is of special interest, i.e., when the distance between the slot edges is several times less than the hole diameter. In this case brittle fracture occurs by spalling of the cylindrical part (a projection that is called a core) and transformation of the body into a pit. The dead-end of the slot is the focus of the stress concentration, sometimes resulting in phenomena of the rock-burst type in hard rocks. Conditions for their origination are clarified, which is important for estimates of the magnitude of mountain pressure. Methods utilized in /1/ are extended in this paper and the results of this research are refined.

1. Derivation of the basic equations.

Under the action of the pressures $p_a(z)$ and $p_b(z)$ on the inner and outer sides of an elastic ring, corresponding radial dispalcements occur.

$$Eu_r(a+0,z) = \frac{2a}{b^2-a^2}\left[\left(\frac{a^2}{4}\nu_- + \frac{b^2}{4}\nu_+\right)p_a - b^2 p_b\right]$$

$$Eu_r(b-0,z) = \frac{2b}{b^2-a^2}\left[a^2 p_a - \left(\frac{b^2}{4}\nu_- + \frac{a^2}{4}\nu_+\right)p_b\right]$$

$$\nu_\pm = 2(1 \pm \nu)$$

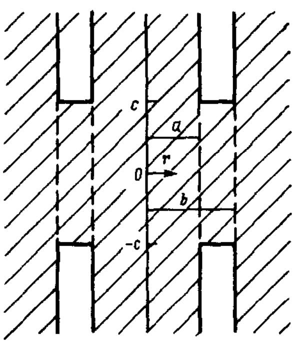

Fig.1

where E is Young's modulus, and ν is Poisson's ratio. On the other hand, the pressure $p_a(z)$ on the surface of an infinite continuous cylinder will cause radial displacements /2/ for $|z| \leqslant c$

$$Eu_r(a-0,z) = -2a(1-\nu^2)\int_{-c}^{c} L_a'(z-s)\,p_a(s)\,ds$$

$$L_a(z) = \frac{1}{\pi}\int_0^\infty G_a(\lambda)\sin\lambda\frac{z}{a}\frac{d\lambda}{\lambda}$$

$$G_a(\lambda) = -\{\nu_- + \lambda^2[1-\varkappa_a^2(\lambda)]\}^{-1},\ \varkappa_a(\lambda) = I_0(\lambda)/I_1(\lambda),$$

($I_{0,1}(\lambda)$ is the modified Bessel function of the first kind).

The condition of continuity for the radial displacements and their derivatives on the inner side of the ring and the surface of the continuous cylinder yields an equation connecting the desired pressures $p_{a,b}(z)$

$$q_a p_a'(z) - p_b'(z) + \varepsilon\int_{-c}^{c} W_a(z-s)\,p_a(s)\,ds = 0,\quad |z| \leqslant c \tag{1.1}$$

and the relationship

$$q_a p_a(0) - p_b(0) + \varepsilon\int_{-c}^{c} L_a'(s)\,p_a(s)\,ds = 0 \tag{1.2}$$

Here

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$$q_a = {}^1/_4\,(\nu_- k^2 + \nu_+), \quad \varepsilon = (1-\nu^2)(1-k^2), \quad k = a/b$$

Utilizing the asymptotic formula

$$G_a(\lambda) = \lambda^{-1} + (1-2\nu)\lambda^{-2} + O(\lambda^{-3}), \quad \lambda \to \infty$$

the kernels $W_a(z)$ and $L_a'(z)$ can be represented in the form of a sum

$$\begin{aligned}
&W_a(z) = W_a^*(z) - \frac{1-2\nu}{2a^2}\,\mathrm{sign}\, z, \\
&L_a'(z) = \frac{1}{\pi a} K_0\left(\nu_+\left|\frac{z}{a}\right|\right) + V_a(z) \\
&W_a^*(z) = -\frac{\nu_+}{\pi a^2} K_1\left(\nu_+\frac{z}{a}\right) - W_a^0(z) \\
&W_a^0(z) = \frac{1}{\pi a^2}\int_0^\infty \left[\lambda G_a(\lambda) - \frac{\lambda}{\sqrt{\lambda^2+\nu_+^2}} - \frac{1-2\nu}{\lambda}\right]\sin\lambda\frac{z}{a}\,d\lambda \\
&V_a(z) = \frac{1}{\pi a}\int_0^\infty \left[G_a(\lambda) - \frac{1}{\sqrt{\lambda^2+\nu_+^2}}\right]\cos\lambda\frac{z}{a}\,d\lambda
\end{aligned} \tag{1.3}$$

where $K_{0,1}(z)$ is a Macdonald function.

Considering now the strain of an unbounded elastic body with a hole of radius b whose axis coincides with the z axis, as before we obtain the equation

$$k^2 p_a'(z) - q_b p_b'(z) - \varepsilon\int_{-c}^{c} W_b(z-s)\,p_b(s)\,ds = 0, \quad |z| \leqslant c \tag{1.4}$$

and the integral relationship

$$k^2 p_a(0) - q_b p_b(0) - \varepsilon\int_{-c}^{c} L_b'(s)\,p_b(s)\,ds = (k^2-1)\,p_\infty \tag{1.5}$$

where $q_b = {}^1/_4\,(\nu_- + k^2\nu_+)$. Here also

$$G_b(\lambda) = \lambda^{-1} - (1-2\nu)\lambda^{-2} + O(\lambda^{-3}), \quad \lambda \to \infty$$

while relationships identical to (1.3), on replacing a by b, $\mathrm{sign}\, z$ by $-\mathrm{sign}\, z$ and ν_+ by ν_-, hold for the functions $W_b(z)$ and $L_b'(z)$.

In particular, for $a = 0$ the integrodifferential equation (IDE)

$$p_b'(z) + \nu_+\int_{-c}^{c} W_b(z-s)\,p_b(s)\,ds = 0, \quad |z| \leqslant c \tag{1.6}$$

and the relationship

$$p_b(0) + \nu_+\int_{-c}^{c} L_b'(s)\,p_b(s)\,ds = \frac{2p_\infty}{1-\nu} \tag{1.7}$$

follows from (1.4) and (1.5) for determining the pressure $p_b(z)$ in the case of a pit.

In the general case $(0 < k < 1)$ relationships (1.4) and (1.1) in conjunction with conditions (1.5) and (1.2) are a system of singular IDE in the pressures $p_{a,b}(z)$ on the boundaries of the separated bodies (the solid cylinder and the unbounded body with the cylindrical hole). We note that a uniform stress state $(p_{a,b}(z) = p_\infty)$ is set up at a sufficient distance from the cavity dead-end for $c \gg b$ and in the limit case when $c = \infty$ the system of equations obtained has the trivial solution $p_{a,b}(z) = p_\infty$.

Integrating (1.1), (1.4) and (1.6) with respect to the variable z and transferring the origin to the point $z = -c$ for $c \gg b$, we arrive at the Wiener-Hopf integral equations obtained in /1/

$$P_b(z) = Q_a(z), \quad k^2 P_a(z) = Q_b(z), \quad z \geqslant 0 \tag{1.8}$$

$$P_b(z) + \nu_+\int_0^\infty L_b'(z-s)\,P_b(s)\,ds = \nu_+ p_\infty L_b^\circ(z) \tag{1.9}$$

$$Q_{a,b}(z) = q_{a,b}P_{a,b}(z) - \varepsilon\left[p_\infty L_{a,b}^0(z) - \int_0^\infty L_{a,b}'(z-s)\,P_{a,b}(s)\,ds\right]$$

$$P_{a,b}(z) = p_{a,b}(z) - p_\infty, \quad L_{a,b}^0(z) = {}^1/_2 G_{a,b}(0) - L_{a,b}(z), \quad z \geqslant 0$$

We note that the method used in /1/ to solve the integral Eqs.(1.8) is not correct since the independent equations obtained for the desired pressures are not convolutions.

2. Solution of the IDE. An exact solution of (1.8) and (1.9) can be constructed that would be expressed in terms of multiple singular integrals of known functions by reduction to a Riemann boundary value problem on an unlimited line. However, such a solution will not be effective from the calculational veiwpoint when going from the transform to the original and the subsequent determination of the state of stress and strain in the neighbourhood of the slot dead-end. An approximate solution of the IDE (1.1) and (1.4) will be constructed here in conjunction with relationships (1.2) and (1.5), and based on application of quadrature formulas by the discrete-vortices method and the simplest finite-difference approximations of the derivatives of the desired functions.

It is seen that the derivatives of the pressures $p_{a,b}(z)$ have a logarithmic singularity on the boundary $z=\pm c$ of the contact zone, and the pressures are themselves bounded everywhere in these zones including the boundary. This property of the solutions of the equations obtained follows from the representations of (1.3), its analogous representations for the functions $W_b(z)$ and $L_b'(z)$ and the asymptotic relationships

$$K_0(x)\sim \ln[2/(\gamma x)],\quad K_1(x)\sim 1/x,\quad x\to 0,\quad \gamma=\text{const}$$

Indeed, by setting $p_{a,b}(s)=p_{a,b}(c)+\varphi_{a,b}(z)$ where $\varphi_{a,b}(z)\to 0$ as $z\to c$ and substituting this expansion into (1.4) and (1.1) we will have

$$q_a p_a'(z)-p_b'(z)=(\pi a)^{-1}\varepsilon p_a(c)\ln(c-z)+\psi_1(z)$$
$$k^2 p_a'(z)-q_b p_b'(z)=-(\pi b)^{-1}\varepsilon p_b(c)\ln(c-z)+\psi_2(z),\quad z\to c,$$

where $\psi_{1,2}(z)$ are functions that are bounded as $z\to c$. Taking into account that the determinant of the system of equations obtained is different from zero, we obtain the assertion required.

We introduce the new functions

$$\Phi_{a,b}(z)=\int_0^z p_{a,b}(s)\,ds,\quad z\geqslant 0\,;$$

then (1.1) and (1.4) and the relationships (1.2) and (1.5) can be rewritten in the form

$$q_a\Phi_a''(z)-\Phi_b''(z)-\varepsilon_0 a^{-2}\Phi_a(z)+\varepsilon A_a(z)=0 \tag{2.1}$$
$$k^2\Phi_a''(z)-q_b\Phi_b''(z)-\varepsilon_0 b^{-2}\Phi_b(z)-\varepsilon A_b(z)=0$$

$$A_{a,b}(z)=\int_0^c[W_{a,b}^*(z-s)+W_{a,b}^*(z+s)]\,\Phi_{a,b}'(s)\,ds,$$
$$\varepsilon_0=\varepsilon(1-2\nu)$$
$$q_a\Phi_a'(0)-\Phi_b'(0)+2\varepsilon B_a=0 \tag{2.2}$$
$$k^2\Phi_a'(0)-q_b\Phi_b'(0)-2\varepsilon B_b=(k^2-1)\,p_\infty$$
$$B_{a,b}=\int_0^c L_{a,b}'(s)\,\Phi_{a,b}'(s)\,ds$$

and moreover

$$\Phi_{a,b}(0)=\Phi_{a,b}''(0)=0 \tag{2.3}$$

We divide the segment of integration $[0,c]$ into $N+1$ equal parts by points $s_j=ct_j$, where $t_j=jh$, $h=(N+1)^{-1}$, $j=0,1,\ldots,N+1$, and use the notation $z_i=ch\,(i-1/2)$, $i=1,2,\ldots,N+1$. Using the quadrature formula for rectangles for the singular and regular integrals /3/

$$\int_0^c W(z_i\pm s)\,\Phi'(s)\approx W(z_i\pm c)\,\Phi(c)-W(z_i)\,\Phi(0)-$$
$$\sum_{k=1}^{N+1}[W(z_i\pm s_k)-W(z_i\pm s_{k-1})]\,\Phi(z_k),\quad i=1,2,\ldots,N+1$$

as well as the finite-difference approximation

$$\Phi''(zi)\approx h^{-2}[\Phi(z_{i+1})-2\Phi(z_i)+\Phi(z_{i-1})],\quad i=2,3,\ldots,N$$
$$\Phi''(z_1)\approx {}^4/_3h^{-2}[\Phi(z_2)-3\Phi(z_1)+2\Phi(0)]$$
$$\Phi''(z_{N+1})\approx {}^4/_3h^{-2}[2\Phi(c)-3\Phi(z_{N+1})+\Phi(z_N)]$$

we reduce (2.1) to a system of $2(N+1)$ linear algebraic equations in $2(N+2)$ unknowns $\Phi_{a,b}(c)$, $\Phi_{a,b}(z_i)$, $i = 1, 2, \ldots, N+1$. These equations, in combination with relationships (2.2), also written in discrete form by using the quadrature and discrete formulas

$$\Phi'(0) \approx 2h^{-1}[\Phi(z_1) - \Phi(0)]$$

$$\int_0^c L'_{a,b}(s)\Phi'(s)\,ds \approx L_{a,b}(z_1)\Phi'(0) + \sum_{k=1}^{N} L'(s_{k+1})[\Phi(z_{k+1}) - \Phi(z_k)] + L'(c)[\Phi(c) - \Phi(z_{N+1})],$$

form a complete system of $2N+4$ equations in the discrete values of the pressures on the boundary of the continuous cylinder and the unbounded body with the cylindrical cavity.

The results of a pressure computation by the method described above are presented in Fig.2a for $\nu = 0.25$, $k = 0.8$, and $\beta = 1$, where $\beta = c/a$ in the case of the cylindrical cavity and $\beta = c/b$ in the case of the pit. The pressure $p_a(z/a)$ on the boundary of the solid cylinder is shown by the dashed line, and the pressure $p_b(z/a)$ on the boundary of the cylindrical hole by the dash-dot line and the pressure $p_b(z/b)$ in the case of the pit by the solid line.

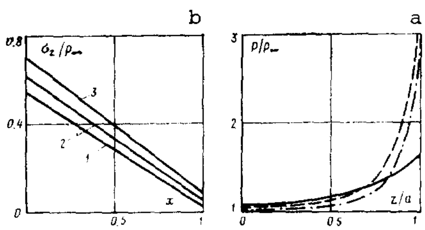

Fig.1

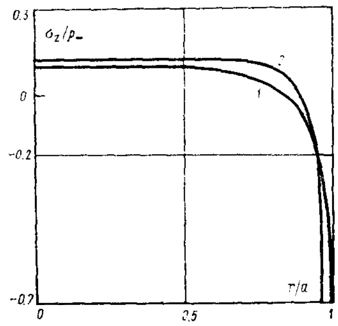

Fig.2

For a known concentration of the pressure $p_a(z)$ on the boundary of the solid cylinder the axial stress σ_z is determined by the integral

$$\sigma_z(r, z) = -\frac{2\beta}{\sqrt{2\pi}} \int_0^\infty \mu G_a(\mu) Q(\mu, r/a) p_a^*(\mu) \cos\mu \frac{z}{a}\,d\mu \tag{2.4}$$

where $p_a^*(\mu)$ is the Fourier transform of the pressure $p_a(z)$

$$p_a^*(\mu) = 2(c\sqrt{2\pi})^{-1} \int_0^c p_a(t)\cos\mu\frac{t}{a}\,dt$$

$$Q(\mu, t) = \mu t I_1(\mu t)/I_1(\mu) + [2 - \mu\varkappa_a(\mu)] I_0(\mu t)/I_1(\mu)$$

A numerical computation using (2.4) shows (Fig.2b) that taking account of the pressure concentration around the cavity dead-end raises the maximum tensile stress σ_z on the boundary of the solid cylinder. Thus, for $\nu = 0.25$ and values of k equal to 0.6, 0.8, 0.9 (lines *1-3* in Fig.2b, respectively, $x = (z - a)/c$) the maximum value of the stress $\sigma_z(a, z)$ varies between $0.71\,p_\infty$ and $0.76\,p_\infty$. Without taking account of the stress concentration $(p_a(z) = p_\infty)$ the maximum magnitude of σ_z is approximately $0.45\,p_\infty$.

A computation of the normal dimensionless stresses $\sigma_z(r, c)/p_\infty$ in the cavity dead-end for values of the parameter k equal to 0.6 and 0.9 (curves *1* and *2* in Fig.3) shows the presence of tensile stresses within the solid cylinder and compressive stresses near its boundary. The difference in the data presented in Figs.2b and 3 is due to the discontinuous nature of the stresses at the singular points $r = a, |z| = c$ of the body boundary and is a result of the model used for the deformation of an interlayer between the surfaces $r = a$ and $r = b$.

The presence of stress concentration in the cavity dead-end results in an increase in the pressure p_∞ to the formation of annular cracks in the sections $|z| = c$ of the solid cylinder with subsequent spalling of the cylindrical part and transformation of the structure into a pit.

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ON TWO MIXED PROBLEMS OF ANTIPLANE STRAIN OF AN ELASTIC WEDGE WITH CIRCULAR HOLES*

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Two problems are examined: 1) a wedge with a circular hole, clamped along the lower face, is subjected to the action of shear forces along the upper face, and 2) a rigid stamp acts on the upper face instead of shear forces. The circular hole is assumed to be load-free. Both problems reduce to a set of infinite systems of linear equations with a completely continuous operator l_2 under the condition that the circle does not touch the sides of the angle. These equations enable the method of reduction to be used. Formulas obtained, that relate the basis solutions of the Laplace equation in two different polar coordinate systems, are utilized in the solution. The method can be extended to the case of a wedge with several circular holes.

The problem of the deformation of a wedge with a circular hole was first examined in one special in /1/, however, the infinite system obtained there remained uninvestigated.

1. We present the relationships between the basis solutions of Laplace's equation in a plane (Figs.1 and 2; $OO_1 = h$, $O_2O_2 = R$), which enable us to change from one system of polar coordinates to another

$$\rho^{-s}e^{is\varphi} = \left(\frac{e^{i\alpha}}{h}\right)^s \Gamma(1-s) \sum_{n=0}^{\infty} \left(\frac{\rho_1}{h}\right)^n \frac{e^{-in\varphi_1}}{n!\,\Gamma(1-s-n)} \quad (\rho_1 < h) \tag{1.1}$$

$$\left(\frac{\rho_1}{h}\right)^{-n} e^{\pm in\omega_{1,2}} = \frac{i}{2(n-1)!} \int\limits_{\Gamma} \frac{\Gamma(s)\, h^s \rho^{-s}}{\sin \pi s\, \Gamma(1+s-n)} e^{\pm is\psi_{1,2}}\, ds \tag{1.2}$$

$$(\Gamma\colon\ 0 < \operatorname{Re} s < 1,\quad s = \alpha + i\tau),\quad 0 \leqslant \varphi_1 < 2\pi$$

$$\psi_1 = \varphi - \pi - \alpha,\quad \omega_1 = \varphi_1,\quad \alpha < \varphi < 2\pi + \alpha$$

$$\psi_2 = -\varphi - \pi + \alpha,\quad \omega_2 = -\varphi_1,\quad -2\pi + \alpha < \varphi < \alpha$$

$$\rho_k^{-n} e^{in\varphi_k} = \sum_{m=0}^{\infty} (-1)^m C_{m+n-1}^m \rho_j^m R^{-(m+n)} \times \tag{1.3}$$

$$\times e^{im\varphi_j + i(m+n)\alpha_{kj}},\quad \rho_j < R;\quad k, j = 1, 2;\quad k \neq j$$

We will apply (1.2) with ω_1 and ψ_1 to satisfy the boundary condition on the face $\varphi = \omega > \alpha$ and with ω_2 and ψ_2 on the face $\varphi = 0 < \alpha$.

Formula (1.1) is obtained as follows. The boundary value problem of finding a harmonic function within a circle of radius $\rho_1 < h$ with centre at the point O_1 (Fig.1) is solved. Values of another harmonic function $\rho^{-s}e^{is\varphi}$ are taken as boundary values. Hence we obtain the equality of the two harmonic functions

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